

Generalization of order separability for free groups.

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Abstract

In this work the author studies the property close to property of order separability.

Key words: free groups, residual properties.

MSC: 20E26, 20E06.

1 Introduction.

Definition 1. We say that a group G is *order separable* if, for each elements g, h of G such that g and $h^{\pm 1}$ are not conjugate, there exists a homomorphism of G onto a finite group such that the orders of the images of g and h are different.

In [1] it is proved that free groups are order separable. This result was generalized in [2] where it was shown that each free group F is omnipotent that is for an arbitrary finite set of elements g_1, \dots, g_n of F such that for each $i \neq j$ elements g_i and g_j have no nontrivial conjugate powers there exists a constant k such that for each ordered sequence of positive integers l_1, \dots, l_n there exists a homomorphism φ of F onto a finite group such that the order of $\varphi(g_i)$ is kl_i .

It is known also that the property of order separability is inherited by free products [3]. The following theorem generalizes the property of order separability for free groups.

Theorem. *Let u_1, u_2 be the elements of a free group F , which does not belong to conjugate cyclic subgroups. Then for each prime number p and for each integer m there exists a homomorphism φ of F onto a finite p -group such that u_1 and u_2 do not belong to the kernel of φ and $|\varphi(u_1)| \mid |\varphi(u_2)| = p^n$.*

2 Notations and definitions.

We shall use the correspondence between the actions of a free group $F(x, y)$ with basis x, y and the graphs satisfying the following properties:

- 1) for each vertex p of a graph there exist exactly one edge with label x that goes away from this vertex and exactly one edge with label y that goes into p ;
- 2) we consider that for each labelled edge there exists the corresponding inverse edge and each two edges with labels are not mutually inverse;

Definition 2. We say that the graph is the *action graph* of a free group $F = F(x, y)$ if it satisfies the properties 1), 2). We shall consider additionally that all labelled edges in this graph are positively oriented and set the orientation of this graph.

If φ is a homomorphism of a group F then the Cayley graph of the group $\varphi(F)$ with the generating set $\{\varphi(x), \varphi(y)\}$ is the action graph for F (we identify labels $\varphi(x), \varphi(y)$ with x, y correspondingly in this graph).

Let Γ be the action graph of the free group F . Fix a natural number n , vertex p of the graph Γ and element $z \in \{x^{\pm 1}, y^{\pm 1}\}$ of the group F . Consider n copies of the graph $\Gamma : \Delta_1, \dots, \Delta_n$. Let p_i be the vertex of the graph Δ_i corresponding to the vertex p of the graph Γ . If $z \in \{x, y\}$ then denote by q_i the vertex such that there exists the edge f_i with label z which goes into q_i from p_i . If $z \in \{x^{-1}, y^{-1}\}$ then denote by q_i the vertex such that there exists the edge f_i with label z^{-1} which goes away from q_i into p_i . We construct the graph $\Delta = \gamma_n(\Gamma; p; z)$ from graphs $\Delta_i, 1 \leq i \leq n$ deleting edges f_i and connecting vertexes p_i and q_{i+1} by an edge whose label equals either z when $z \in \{x, y\}$ or z^{-1} when $z \in \{x^{-1}, y^{-1}\}$ (indices are modulo n). If $z \in \{x, y\}$ then this new edge goes away from the vertex p_i . If $z \in \{x^{-1}, y^{-1}\}$ then this new edge goes into the vertex p_i . The graph Δ is the action graph of the group F .

If S is a path in a graph then $\alpha(S), \omega(S)$ are the beginning and the end of S correspondingly. If $S = e_1 \dots e_n$ is a path in the action graph of the group F , then $\text{Lab}(S) = \text{Lab}(e_1)' \dots \text{Lab}(e_n)'$ is the label of the path S , where $\text{Lab}(e_i)' = \text{Lab}(e_i)$ is a label of edge e_i in case e_i is positively oriented and if the edge e_i is negatively oriented then $\text{Lab}(e_i)' = \text{Lab}(e_i')^{-1}$, where edges e_i, e_i' are mutually inverse.

Definition 3. Suppose we have two action graphs of the group F — Γ and $\gamma_n(\Gamma; q; z)$. If p is a vertex of Γ then p^i is a vertex in the graph $\gamma_n(\Gamma; q; z)$ which belongs to i -th copy of the graph Γ and corresponds to the vertex p . If S is a path in Γ then S^i is the path in $\gamma_n(\Gamma; q; z)$ which goes from the vertex $\alpha(S)^i$ and whose label is equal to the label of the path S .

Definition 4. Let u be a cyclically reduced element of the free group F . If $S = e_1 \dots e_n$ is a closed path without returnings whose label equals u^k in the action graph of F then we say that S is the u -cycle whose length equals k (we consider that there exists exactly one subpath of the path S ending in $\omega(S)$ whose label equals u).

3 Proof of theorem.

Fix elements u_1, u_2 of F . We consider that these elements are cyclically reduced.

Suppose that $u_i = u_i^{p^{m_i} t_i}, i = 1, 2$, where $p \nmid t_i$. Denote by v_i the elements $u_i^{t_i}, i = 1, 2$. Without loss of generality we may assume that $m_1 \geq m_2$. If there exists a homomorphism ψ of F onto a finite group such that $|\psi(v_1)| \mid |\psi(v_2)| = p^{n+m_1-m_2}$ then $|\psi(u_1)| \mid |\psi(u_2)| = p^n$. Hence we may assume that the subgroups $\langle u_1 \rangle$ and $\langle u_2 \rangle$ are p' -isolated.

There exists a homomorphism φ of F onto a finite p -group such that the images of the elements u_1, u_2 are nonunit p -elements [4]. Since all p' -isolated cyclic subgroups of free groups are separable in the class of finite p -groups we may also assume that all u_1 - and u_2 -cycles in the Cayley graph Γ of the group $\varphi(G)$ with the set of generators $x = \varphi(x'), y = \varphi(y')$ are simple. Without loss of generality we may consider that $|\varphi(u_1)| = p^k, |\varphi(u_2)| = p^{k+l}, k \geq 1$. Put $\Gamma_{-1} = \Gamma$. Let $u_1 = y_0^{\varepsilon_1} \dots y_{k-1}^{\varepsilon_1}, y_i \in \{x, y\}, \varepsilon_i \in \{-1, 1\}$ be the reduced form of the element u_1 in the basis x, y . Fix an arbitrary vertex q in the graph Γ_{-1} .

For $i > -1$ we shall define by the induction the graph $\Gamma_i = \gamma_p(\Gamma_{i-1}; q_{i-1}; y_{i'}^{\varepsilon_{i'}})$ and the path S_i in Γ_i whose length equals $i + 1$, where i' is the remainder from a division of i on k . If $i > -1$ then q_i is the vertex of the graph Γ_i which is the end point of the path S_i . The vertex q_{-1} is equal to the vertex q . If $i > -1$ we define a path S_i in the graph Γ_i in the following way. If $i = 0$ then S_0 is the first edge of the u_1 -cycle which goes from the vertex q_{-1}^1 or its inverse. In case $i > 0$ we define the path S_i as $S_{i-1}^1 \cup f_i$ where f_i is the edge of the graph Γ_i one of whose endpoints coincides with $\omega(S_{i-1}^1)$ and the label of f_i or its inverse equals $y_{i'}$. Also if $\varepsilon_i = 1$ then f_i is positively oriented and has a label. If $\varepsilon_i = -1$ then f_i is negatively oriented. It is easy to notice that the length of each u_1 - or u_2 -cycle in Γ_i is the power of p . Since all u_1 - and u_2 -cycles are simple in the graph Γ_{-1} this condition is also held for Γ_i for each i . Besides for each i the graph Γ_i contains the maximal u_1 -cycle which contains the path S_i .

Suppose that there exists i such that the following conditions are true. In the graph Γ_j each maximal u_2 -cycle contains the path S_j for all $j \leq i$ but not for $j = i + 1$ (if $i + 1 = 0$ we simply consider that in the graph Γ_0 not all maximal u_2 -cycles contains S_0). Notice that if $j \leq i$ then in the graph Γ_j the length of the maximal u_i -cycle coincide with $|\varphi(u_i)|p^{j+1}$, $i = 1, 2$. In the graph Γ_{i+1} the length of the maximal u_1 -cycle equals $|\varphi(u_1)|p^{i+1}$. Lets find the length of the maximal u_2 -cycle in Γ_{i+1} . Fix a vertex r of Γ_i . Then the length of the u_2 -cycle which goes away from r is equal to $|\varphi(u_2)|p^k$, $k \leq i$. Then for each $l = 1, \dots, p$ in the graph Γ_{i+1} the length of the u_2 -cycle that goes away from the vertex r_l is not more than $|\varphi(u_2)|p^{k+1}$. Suppose that the u_2 -cycle T of Γ_i starting from r is maximal, that is its length coincide with $|\varphi(u_2)|p^i$. The path S_i is contained in T . The u_2 -cycle T^1 of Γ_{i+1} contains the path S_i^1 . From the condition on i and the simplicity of u_2 -cycles of Γ_{i+1} we may deduce that T^1 does not contain the edge f_{i+1} . This u_2 -cycle does not also contain the edge connecting the first and the last copies of the graph Γ_i because otherwise T would have the self-intersection in the vertex $\omega(S_{i-1})^1$. Hence the length of T^1 coincide with the length of T . It means that the length of the maximal u_2 -cycle in Γ_{i+1} coincide with $|\varphi(u_2)|p^i$. Since u_1 and u_2 does not belong to the conjugate cyclic subgroups then there exists i that satisfies the conditions mentioned above.

The number of vertices in the graph Γ_{i+1} equals $|\varphi(F)|p^{i+2}$. So there exists a homomorphism φ_1 of F onto a finite p -group such that in the Cayley graph of the group $\varphi_1(F)$ with the set of generators $\varphi_1(x'), \varphi_1(y')$ all u_1 - and u_2 -cycles are simple. Besides $|\varphi_1(u_2)|/|\varphi_1(u_1)| = |\varphi(u_2)|/|\varphi(u_1)| * 1/p$.

If we construct graphs Γ_i using the element u_2 instead of u_1 we obtain the homomorphism φ_2 that satisfies the same conditions concerning to the u_1 - and u_2 -cycles in the graph $\varphi_2(F)$ and the homomorphism φ_1 and $|\varphi_2(u_2)|/|\varphi_2(u_1)| = |\varphi(u_2)|/|\varphi(u_1)| * p$. In order to obtain the homomorphism of F onto a finite p -group such that the ratio of orders of the images of u_1 and u_2 sufficient p^n it is enough to apply these procedures several times. The theorem is proved.

Acknowledgements.

The author thanks A. A. Klyachko for valuable comments and discussions.

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